THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2016–2017)
Introduction to Topology
Exercise 12 Fundamental group

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let $\alpha, \beta, \gamma\colon\ I = [0,1] \to X$ be paths with $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. Show that $H\colon\ I \times [0,1] \to X$ is a homotopy rel $\{0,1\}$ for $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$ where

$$H(s,t) = \begin{cases} \alpha\left(\frac{s}{a(t)}\right) & s \in [0, a(t)] \\ \beta\left(\frac{s-a(t)}{b(t)-a(t)}\right) & s \in [a(t), b(t)] \\ \gamma\left(\frac{s-b(t)}{1-b(t)}\right) & s \in [b(t), 1]\ , \end{cases}$$

for any continuous functions $1/4 \le a(t) < b(t) \le 3/4$ with $a(0) = 1/4$, $a(1) = 1/2$, $b(0) = 1/2$, $b(1) = 3/4$.

Furthermore, give an explicit example of $a(t)$ and $b(t)$.

2. Let α be a loop in X based at $x_0 \in X$ and c be the constant path at x_0. Give explicit homotopies to show that

$$\alpha * c \simeq \alpha \simeq c * \alpha \quad \text{rel} \quad \{0,1\}.$$

Do you have a similar statement when α is not a loop?

3. Prove that inverse exists in $\pi_1(X, x_0)$ with product $*$.

4. Let X be a path connected space and $x_0, x_1 \in X$. Find an isomorphism between the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

5. Use winding number argument to show that $\pi_1(\mathbb{S}^1) = \pi_1(\mathbb{C} \setminus \{0\}) = (\mathbb{Z}, +)$.

6. Formulate the major steps in proving every loop in the torus is a product of the two generating loops, α and β.

7. Formulate the major steps in proving every loop in the projective plane is a product of the generating loops.

8. Give the outline of an argument that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

9. Let X be a topological space and $\alpha_1, \ldots, \alpha_n$ are loops based at $x_0 \in X$. Show that if $[\alpha_1] \cdots [\alpha_n] = 1$ in $\pi_1(X, x_0)$, then the loop $\alpha_1 * \cdots * \alpha_n$ is homotopic to a loop γ at x_0 such that γ is the boundary of a disk in X.

The questions below involves a concept call *homotopy equivalences*. Let X, Y be two spaces. Two mappings $f:\ X \to Y$ and $g:\ Y \to X$ are called *homotopy equivalences* between X and Y; and the spaces X and Y are called *homotopy equivalent* or *of the same homotopy type* if

$$g \circ f \simeq \mathrm{id}_X \quad \text{and} \quad f \circ g \simeq \mathrm{id}_Y \ .$$

The mappings f and g are called *homotopy inverses*.

5. (a) Let X, Y_1, and Y_2 are topological spaces such that $Y_1 \simeq Y_2$ (they are homotopy equivalent). Then the set of homotopy classes $[X, Y_1]$ and $[X, Y_2]$ are bijective.

   (b) Let X_1, X_2, and Y are topological spaces such that $X_1 \simeq X_2$ (they are homotopy equivalent). Then the set of homotopy classes $[X_1, Y]$ and $[X_2, Y]$ are bijective.

6. Show that a space of homotopy type of a point if and only if it is contractible.

7. Let X be a topological space and Y be contractible. Prove that X and $X \times Y$ are homotopy equivalent. If $y_0 \in Y$ then $X \times \{y_0\} \simeq X \times Y$ where the homotopy equivalences can be chosen rel $X \times \{y_0\}$.

8. Show that $\mathbb{S}^1$ and $\mathbb{C} \setminus \{0\}$ are homotopy equivalent.

9. Show that if X and Y are of the same homotopy type, then $\pi_1(X) = \pi_1(Y)$.